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Fidelity for States of Spin-$\frac{1}{2}$ Particles Moving in a Traversable Wormhole Spacetime

B. Nasr Esfahani

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Abstract Fidelity for states of spin-$\frac{1}{2}$ particles moving in a static spherically symmetric traversable wormhole spacetime is discussed. When the centroid of the corresponding wave packet moves along a specified path in the gravitational field, both acceleration and gravity cause to transform the state of the particle. For circular orbits of the centroid coinciding the throat of wormhole, the fidelity between initial and final states of the whole system as well as the fidelity of the spin parts of the states are equal to the unity. This means that, the error in quantum communication diminishes on such a paths. For fixed elapsed proper time and angular momentum of the centroid, there always exists one circular orbit with determined radius on which the fidelity of spin parts is minimum. The fidelity for wave packets moving along a radial geodesic toward the throat of wormhole is also discussed. In this case, the centroid traverses the wormhole and reaches to the other side, with a perfect fidelity for the spin parts, though the fidelity for the states of the whole system is not perfect.

Keywords Local inertial frame · Wigner rotation · Density matrix · Fidelity · Wormhole spacetime

1 Introduction

Relativistic quantum information theory may become a necessary theory in the near future. Therefore, it is important to study all those processes that might affect quantum entanglement. Entanglement is a strange feature of quantum theory and leads to a nonlocal correlation called the Einstein-Podolsky-Rozen (EPR) correlation [3]. The entangled subsystems are correlated beyond what is classically possible [4]. It is this feature that enables quantum communication. Recently, a number of papers [1, 5, 9, 14] have discussed how entanglement is affected by the Lorentz transformation in the jargon of special relativity.

B.N. Esfahani (✉)
Department of Physics, Faculty of Sciences, University of Isfahan, Isfahan 81744, Iran
e-mail: ba_nasre@sci.ui.ac.ir

The question of how entanglement is affected by a gravitational field is discussed by Terashima and Ueda by extending the special relativistic considerations to general relativity [18]. They have discussed a mechanism of spin decoherence caused by spacetime curvature for spin-$\frac{1}{2}$ particles moving in a gravitational field. Also, the spin entropy production for particles with arbitrary spin moving in a general static spherically symmetric spacetimes is discussed. In general relativity, a gravitational field represented by spacetime curvature, cause to break the global rotational symmetry. Therefore, the spin in general relativity can be defined only locally by invoking the rotational symmetry of the local inertial frame. Consequently, the motion of the particle in a curved spacetime is accompanied by a continues succession of Lorentz transformations [19]. It is shown that this effect gives rise to a spin entropy production that is unique to general relativity. This means that even if the state of the particle is pure at one spacetime point, it becomes mixed at another spacetime point. The gravitational spin entropy production for particles with arbitrary spin moving in a curved spacetime is also discussed [13].

In this paper which can be considered as an extension of our previous work [13], we discuss the fidelity between the initial and final states of spin-$\frac{1}{2}$ particles moving in a static traversable wormhole spacetime. In Sects. 2 and 3, based on that paper, we first review the subject of spin decoherence for particles moving in a curved spacetime and then apply the results for a traversable wormhole spacetime. Wormholes have a non-trivial topology and can connect two different regions of the Universe. Thus, the study of communication through the throat of wormholes can be important. Note that we have considered *traversable* wormholes that have no event horizon and there is no singularity inside them. Thus, they allow two-way passages of photons and massive particles through them [10]. Suppose that a member of an entangled pair falls into the horizon of a black hole. How is such a state describe by quantum theory? Is the correlation observable? There is no clear answer to this problem [15]. However, there is not such a problem for wormholes. Falling particles traverse the throat, keeping their correlations with particles remaining outside. These results are interesting enough to motivate us consider wormhole spacetimes as an alternative example for discussing the quantum communication in curved spacetimes. We will consider circular and radial paths for the centroid, separately.

There is no experimental evidence for wormholes, and theoretically they suffer the problem of exotic matter that is required to support them [10]. However, based on the recent astrophysical observations, it is generally accepted that the Universe at the present is expanding with acceleration. To explain such the cosmological behavior in the framework of general relativity, we require to suppose that approximately 70% of the Universe is composed of dark energy. There is a substantial body of literature on this issue that the problem of existing of wormholes links to the problem of dark energy [12, 17].

2 Transformation of State of a Particle Moving in a Gravitational Field

First of all we need to define the spin in a gravitational field described by a metric $g_{\mu\nu}(x)$. However, in a curved spacetime the curvature causes to break the global rotational symmetry. Therefore, the spin in general relativity can be defined only locally by switching to an inertial frame at each point and then invoking the rotational symmetry of the local inertial frame.

As is well known, a local inertial frame at each point of spacetime is introduced through a tetrad $e_a{}^\mu(x)$ defined by

$$e_a{}^\mu(x)e_b{}^\nu(x)g_{\mu\nu}(x) = \eta_{ab}, \tag{1}$$

where $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric [11] and Latin indices run over the four inertial-coordinate labels 0, 1, 2, 3, while, Greek letters run over the four general-coordinate labels. Here, a particle is specified by the tetrad $e_0{}^\mu$, which relates the local time to a global time. Of course, since we cannot uniquely choose the time coordinate to define the positive energy, the definition of a particle is not unique [2]. Now, a particle with spin $\frac{1}{2}$ in the curved spacetime is defined as a particle whose one-particle states furnish the spin-$\frac{1}{2}$ representation of the local Lorentz group.

The one-particle momentum eigenstate is described as $|p,\sigma\rangle$ where $p = (\sqrt{|\mathbf{p}|^2 + m^2c^2}, \mathbf{p})$ is the four momentum of the particle as measured in the local inertial frame and $\sigma = \pm\frac{1}{2}$ denotes the z-components of the spin. Then a pure state for the particle with *positive energy*, has the form

$$|\psi\rangle = \sum_\sigma \int d^3 p C_\sigma(p)|p,\sigma\rangle, \tag{2}$$

where amplitudes $C_\sigma(p)$ determine the admixture of the one particle momentum eigenstates in the wave packet [8]. Normalizing $|\psi\rangle$ to unity implies

$$\sum_\sigma \int d^3 p |C_\sigma(p)|^2 = 1, \tag{3}$$

provided that $\langle p', \sigma'|p,\sigma\rangle = \delta^3(\mathbf{p}' - \mathbf{p})\delta_{\sigma'\sigma}$.

Now, consider that the centroid of the wave packet corresponding to (2) is located at point x^μ and is moving with a four-momentum $q^a(x) = e^a{}_\mu(x)(m dx^\mu/d\tau)$ as measured in the local inertial frame at the point x^μ. It is required to assume that the spacetime curvature does not change drastically within the spacetime scale of the wave packet that describes a state of the particle. For forced motions of the centroid there exists an acceleration

$$a^a(x) = e^a{}_\mu(x)(u^\nu(x)\nabla_\nu u^\mu(x)), \tag{4}$$

measured in the local inertial frame. While, for geodesic motions $a^a(x) = 0$.

After an infinitesimal proper time $d\tau$, the centroid moves to a new point $x'^\mu = x^\mu + u^\mu(x)d\tau$. Then, the wave packet is described by a local inertial frame at the new point x'^μ. In the new local inertial frame, the momentum of the centroid changes to $q^a(x') = q^a(x) + \lambda^a{}_b(x)q^b(x)d\tau$, where

$$\lambda^a{}_b(x) = \zeta^a{}_b(x) + \kappa^a{}_b(x), \tag{5}$$

where

$$\zeta^a{}_b(x) = -\frac{1}{mc^2}[a^a(x)q_b(x) - q^a(x)a_b(x)], \tag{6}$$

is the acceleration related part, existing even in special relativity, and

$$\kappa^a{}_b(x) = u^\mu(x)(e_b{}^\nu(x)\nabla_\mu e^a{}_\nu(x)), \tag{7}$$

is the curvature related part, that is the change in the local inertial frame along the path.

It must be noted that $\lambda^a{}_b$ given by (5) satisfies the condition $\lambda_{ab} = -\lambda_{ba}$, then it can describe a local Lorentz transformation $\Lambda^a{}_b(x)$ which is represented infinitesimally as

$\Lambda^a{}_b(x) = \delta^a{}_b + \lambda^a{}_b(x)d\tau$. Corresponding to $\Lambda^a{}_b(x)$, there exists a unitary operator, denoted by $U(\Lambda^a{}_b(x))$, which transforms the momentum eigenstate as

$$|p,\sigma\rangle \to U(\Lambda(x))|p,\sigma\rangle. \tag{8}$$

It can be shown that the state $U(\Lambda)|p,\sigma\rangle$ is also the eigenstate of the momentum operator but with the eigenvalue Λp [20] and we can write at every point

$$U(\Lambda(x)|p,\sigma\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda(x),p))|\Lambda p,\sigma'\rangle, \tag{9}$$

where $W^a{}_b(\Lambda(x),p)$ is the Wigner rotation and $D_{\sigma'\sigma}(W(\Lambda(x),p))$ is its spin-$\frac{1}{2}$ irreducible representation. The infinitesimal Wigner rotation is represented as

$$W^a{}_b(\Lambda(x),p) = \delta^a{}_b + \vartheta^a{}_b d\tau, \tag{10}$$

where $\vartheta^0{}_0(x) = \vartheta^0{}_i(x) = \vartheta^i{}_0(x) = 0$ and

$$\vartheta^i{}_k(x) = \lambda^i{}_k(x) + \frac{\lambda^i{}_0(x)p_k(x) - \lambda_{k0}(x)p^i(x)}{p^0(x)+mc}, \tag{11}$$

where i and k run over the three spatial inertial frame labels $(1,2,3)$ [18]. Note that $\vartheta_{ii} = 0$. The corresponding spin-$\frac{1}{2}$ representation of (10) is

$$\begin{aligned} D_{\sigma'\sigma}(W(\Lambda(x),p)) = \delta_{\sigma'\sigma} &+ i\vartheta_{23}(x)(J_1)_{\sigma'\sigma}d\tau + i\vartheta_{31}(x)(J_2)_{\sigma'\sigma}d\tau \\ &+ i\vartheta_{12}(x)(J_3)_{\sigma'\sigma}d\tau, \end{aligned} \tag{12}$$

where $\{J_1, J_2, J_3\}$ are the generators of the rotation group in a 2-dimensional representation [20].

When the centroid moves along a path $x^\mu(\tau)$ from $x_i^\mu = x^\mu(\tau_i)$ to $x_f^\mu = x^\mu(\tau_f)$, the motion of the wave packet is accompanied by a Lorentz transformation given by

$$\Lambda(x_f,x_i) = T\exp\left[\int_{x_i}^{x_f}\lambda(x(\tau))\,d\tau\right], \tag{13}$$

and, a Wigner rotation as

$$W(\Lambda(x_f,x_i),p) = T\exp\left[\int_{x_i}^{x_f} w(x(\tau))\,d\tau\right], \tag{14}$$

where T is the time-ordering operator and λ and w are matrices that their components are given by (5) and (11), respectively.

Suppose that in the local inertial frame at the initial point x_i^μ, the state is denoted by $|\psi^{(i)}\rangle$ which is given by (2). Then, in the local inertial frame at the final point x_f^μ, the state will be

$$\begin{aligned} |\psi^{(f)}\rangle = U(\Lambda(x_f,x_i))|\psi^{(i)}\rangle = \sum_\sigma\sum_{\sigma'}\int d^3p\sqrt{\frac{(\Lambda p)^0}{p^0}}C_\sigma(p^a) \\ \times D_{\sigma'\sigma}(W(\Lambda(x_f,x_i),p))|\Lambda p,\sigma'\rangle, \end{aligned} \tag{15}$$

which naturally is a pure state.

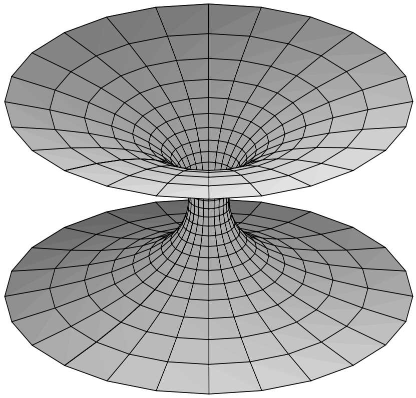

Fig. 1 An embedding surface for a traversable wormhole. Each *circle* denotes actually a sphere and the *circle* of minimum radius denotes the throat surface. The generator *lines* indeed indicate the radial geodesics

3 Wigner Rotation for Spin-$\frac{1}{2}$ Particles Moving in a Traversable Wormhole Spacetime

Here we consider the background geometry to be a traversable wormhole described by the metric

$$ds^2 = -c^2 e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - \frac{B(r)}{r}} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{16}$$

where $B(r)$ and $\Phi(r)$ are called shape function a red shift function, respectively. The shape function controls the shape of wormholes as viewed in an embedding diagram. Moreover, the equation $B(r_0) = r_0$ determines the radius of the throat that is the minimum or lower limit of the radial coordinate, that is $r \geq r_0$. Then, traversable wormholes have no event horizon. Figure 1 shows the embedding diagram for a traversable spherical symmetric wormhole. The circle with the minimum radius denotes the throat which actually is a spherical surface. The generator lines in Fig. 1 indicate radial geodesics and imply that two-way passages through the throat are possible.

The asymptotic flatness requirement for the metric (16) requires that $\lim_{r\to\infty} \frac{B(r)}{r} = \lim_{r\to\infty} \Phi(r) = 0$. The red-shift function $\Phi(r)$ is chosen to be finite everywhere, thus there is no surface of infinite red shift.

We require to introduce a local inertial frame at each point in the spacetime (16). Therefore, we choose the tetrad defined in (1) as

$$e_0{}^t = e^{-\Phi(r)}, \qquad e_1{}^r = \sqrt{1 - \frac{B(r)}{r}}, \qquad e_2{}^\theta = \frac{1}{r}, \qquad e_3{}^\phi = \frac{1}{r\sin\theta}, \tag{17}$$

with all the other components being zero.

In this section we find the Lorentz transformation $\Lambda(x_f, x_i)$ as well as the Wigner rotation $W(\Lambda(x_f, x_i), p)$ for spin-$\frac{1}{2}$ particles moving in a wormhole spacetime described by (16). Since a general motion of the centroid can be considered as a combination of circular and radial motions, we argue on these motions, separately.

3.1 Circular Motion around the Throat

Suppose that the centroid of the wave packet is moving with a constant speed v on a circle with radius r around the throat of wormhole. Regarding the spherical symmetry of the metric, we can choose the plane of motion to be the equatorial plane $\theta = \frac{\pi}{2}$. Thus, the path can be one of the circles shown in Fig. 1. Since v is measured by an observer in the local inertial frame, who uses inertial coordinate labels (0, 1, 2, 3), we can write

$$v = c\frac{dx^3}{dx^0} = c\frac{e^3{}_\phi d\phi}{e^0{}_t c dt} = re^{-\Phi(r)}\frac{d\phi}{dt}. \tag{18}$$

Then, the components of the four-momentum of the centroid in the local inertial frame at any point are

$$q^0 = \gamma mc, \qquad q^1 = q^2 = 0, \qquad q^3 = \gamma mv. \tag{19}$$

Moreover, after some manipulation we see that in this case, the acceleration (20) has only one non-zero component

$$a^1(r) = c^2\gamma^2\sqrt{1-\frac{B(r)}{r}}\left(\Phi' - \frac{1}{r}\frac{\gamma^2-1}{\gamma^2}\right) \tag{20}$$

which lead to the following non-zero components for $\zeta^a{}_b$,

$$\begin{aligned} \zeta^0{}_1 = \zeta^1{}_0 &= c\gamma^3\sqrt{1-\frac{B(r)}{r}}\left(\Phi' - \frac{1}{r}\frac{\gamma^2-1}{\gamma^2}\right), \\ \zeta^1{}_3 = -\zeta^3{}_1 &= -v\gamma^3\sqrt{1-\frac{B(r)}{r}}\left(\Phi' - \frac{1}{r}\frac{\gamma^2-1}{\gamma^2}\right). \end{aligned} \tag{21}$$

These give rise to a generalized Thomas precession [18] in the considered spacetime.

On the other hand, applying (7) for the tetrad (17), we obtain the following non-zero components for $\kappa^a{}_b(x)$,

$$\kappa^0{}_1(r) = \kappa^0{}_1(r) = -c\gamma\Phi'(r)\sqrt{1-\frac{B(r)}{r}}, \tag{22}$$

and

$$\kappa^1{}_3(r) = -\kappa^3{}_1(r) = \gamma\frac{v}{r}\sqrt{1-\frac{B(r)}{r}}. \tag{23}$$

Then, regarding (5), we see that $\lambda^a{}_b(x)$ has only four non-zero components as

$$\lambda^1{}_0(r) = \lambda^0{}_1(r) = \left(\Phi'(r) - \frac{1}{r}\right)\gamma(\gamma^2-1)c\sqrt{1-\frac{B(r)}{r}}, \tag{24}$$

$$\lambda^1{}_3(r) = -\lambda^3{}_1(r) = -\left(\Phi'(r) - \frac{1}{r}\right)\gamma^3 v\sqrt{1-\frac{B(r)}{r}}. \tag{25}$$

These components as substituted in (11) lead to the following non-zero components for $\vartheta^a{}_b$

$$\begin{aligned} \vartheta^1{}_3 = -\vartheta^3{}_1 = &-c\left(\Phi'(r) - \frac{1}{r}\right)\sqrt{1-\frac{B(r)}{r}} \\ &\times\left[q(q^2+1) - q^2\sqrt{q^2+1}\frac{p}{\sqrt{p^2+1}+1}\right], \end{aligned} \tag{26}$$

where $p = \frac{|\mathbf{p}|}{mc}$ and $q = \frac{q^3}{mc}$. Then the infinitesimal Wigner rotation (12), becomes

$$D_{\sigma'\sigma}(W(\Lambda)) = \delta_{\sigma'\sigma} + i\vartheta(J_2)_{\sigma'\sigma} d\tau, \tag{27}$$

where $\vartheta = \vartheta_{31} = -\vartheta^1{}_3$. Correspondingly, the finite Wigner rotation becomes

$$W(\Lambda(\tau_f, \tau_i)) = T \exp\left[\int_{\tau_i}^{\tau_f} \vartheta J_2 d\tau\right]. \tag{28}$$

This readily gives

$$W(\Lambda(\tau)) = \exp[i\vartheta\tau J_2], \tag{29}$$

where $\tau = \tau_f - \tau_i$. Note that time ordering can easily be done since ϑ is time independent. The elements of $D(W(\Lambda(\tau)), p)$ can be obtained by using the Wigner's formula [16]. We have

$$D = \begin{pmatrix} \cos\frac{\Theta}{2} & \sin\frac{\Theta}{2} \\ -\sin\frac{\Theta}{2} & \cos\frac{\Theta}{2} \end{pmatrix} \tag{30}$$

where

$$\Theta = -c\tau\left(\Phi'(r) - \frac{1}{r}\right)\sqrt{1 - \frac{B(r)}{r}} \times \left[q(q^2+1) - q^2\sqrt{q^2+1}\frac{p}{\sqrt{p^2+1}+1}\right], \tag{31}$$

which apparently vanishes at the throat where $B(r) = r$.

3.2 Radial Motion

Now let us consider that the centroid of the wave packet moves with a velocity v along a radial geodesic, say one of the generator lines in Fig. 1. As the figure indicates, in this situation the centroid can traverse the throat. Since v is measured by an observer in the local inertial frame, we can write

$$v = c\frac{dx^1}{dx^0} = c\frac{e^1{}_r dr}{e^0{}_t c dt} = \frac{e^{-\Phi(r)}}{\sqrt{1 - \frac{B(r)}{r}}}\frac{dr}{dt}. \tag{32}$$

Then, we obtain in this case

$$q^0 = \gamma mc, \qquad q^1 = \gamma m v, \qquad q^2 = q^3 = 0, \tag{33}$$

as the components of the four-momentum of the centroid.

Since the centroid moves along a geodesic, $a^a(x) = 0$ and then according to (6) the acceleration related part of (5) vanishes. However, after doing some manipulations we see that the curvature related part $\kappa^a{}_b$ has two non-zero components as

$$\lambda^0{}_1(r) = \lambda^1{}_0(r) = -\gamma c\Phi'(r)\sqrt{1 - \frac{B(r)}{r}}, \tag{34}$$

which consists of a boost along the 1-axis. These elements as substituted in (13) constitute a finite Lorentz transformation Λ. Of course, one should be care of time ordering, since in this case r is a function of τ via radial timelike geodesics. However, note that for wormholes with constant red shift function, $\Phi' = 0$, and so Λ becomes the unit transformation.

Applying (34) in (11), we conclude that all of the components of $\vartheta^i{}_k$ vanish and the Wigner rotation simply becomes $W^a{}_b = \delta^a{}_b$, which has a trivial spin-$\frac{1}{2}$ representation as $D_{\sigma'\sigma}(W(x)) = \delta_{\sigma'\sigma}$.

4 Fidelity of the Initial and Final States

Suppose that in wormhole spacetime of previous section $|\psi^{(i)}\rangle$ shows the state in a local inertial frame located at x^i and $|\psi^{(f)}\rangle$ is the state in the local frame at final point x^f. Here it is convenient to discuss the fidelity $\mathcal{F}$ for these initial and final states. The concept of fidelity is a basic ingredient in communication theory. For any given communication scheme the fidelity is a quantitative measure of the accuracy of the transmission. Fidelity ranges over the interval [0, 1] such that $\mathcal{F} = 1$ indicates a perfect transmission. So we refer to $\mathcal{F} = 1$ as perfect fidelity. In the following we first discuss the fidelity for the states of the whole system which we call it the total fidelity, denoted by $\mathcal{F}_t$. Then, taking the trace over the momentum, we argue on the fidelity for the spin part of the state which we call it the spin fidelity indicated by $\mathcal{F}_s$.

4.1 Total Fidelity

Both of the initial and the final states of the whole system are pure and to calculate the total fidelity we apply the relation

$$\mathcal{F}_t = \langle\psi^{(i)}|\psi^{(f)}\rangle\langle\psi^{(f)}|\psi^{(i)}\rangle. \tag{35}$$

Substituting from (2) and (15), we obtain

$$\begin{aligned}\mathcal{F}_t = &\int d^3p\sqrt{\frac{(\Lambda p)^0}{p^0}}\\ &\times\int d^3p'\sqrt{\frac{(\Lambda p')^0}{p'^0}}C^\dagger(\Lambda p)D(W,p)C(p)[D(W,p')C(p')]^\dagger C(\Lambda p'),\end{aligned} \tag{36}$$

where $D = \begin{pmatrix} D_{\uparrow\uparrow} & D_{\uparrow\downarrow} \\ D^*_{\uparrow\downarrow} & D_{\downarrow\downarrow}\end{pmatrix}$ and $C = \begin{pmatrix} C_\uparrow \\ C_\downarrow\end{pmatrix}$.

For the circular motion around the throat, $D(W, p)$ is given by (30). Moreover, the required Lorentz transformation Λ can be obtained in this case by substituting the elements of the matrix λ from (24) and (25) in (13). Since these elements are independent of proper time, the time ordering can readily be done. Then employing the generators of the Lorentz group [7], and after doing some manipulation, we obtain

$$\Lambda(\tau) = \exp\left[\begin{pmatrix} 0 & \xi\tau & 0 & 0 \\ \xi\tau & 0 & 0 & \omega\tau \\ 0 & 0 & 0 & 0 \\ 0 & -\omega\tau & 0 & 0\end{pmatrix}\right], \tag{37}$$

where $\xi = \lambda^1{}_0$ and $\omega = \lambda^1{}_3$. Now it is important to note that on circular orbits coinciding the throat of wormhole, where $B(r) = r$, the parameters ξ, λ and Θ all vanish. Thus $\Lambda(\tau)$ as well as $D(W, p)$ reduce to their corresponding unit matrices and, as we see from (36), the total fidelity equals the unity implying the possibility of a perfect communication between observers situated at the throat. Evidently, the calculation of total fidelity for an arbitrary circular orbit is a complicated task that we do not follow here. Instead, in the next subsection we will do this for the fidelity of the spin parts of the system.

For the radial motion as we discussed in Sect. 3.2, the Wigner rotation is trivial , however there exists a Lorentz transformation. Thus we have a total fidelity as

$$\mathcal{F}_t = \int d^3p \sqrt{\frac{(\Lambda p)^0}{p^0}} \int d^3p' \sqrt{\frac{(\Lambda p')^0}{p'^0}} C^\dagger(\Lambda p) C(p) C(p')^\dagger C(\Lambda p'), \tag{38}$$

where the Lorentz transformation Λ can be determined by using the elements given in (34). Recall that for constant red shift function wormholes Λ reduces to the unit transformation and the fidelity (38) becomes perfect.

4.2 Spin Fidelity

Corresponding to the initial state $|\psi^{(i)}\rangle$ given by (2), there exists a density matrix as $\rho^{(i)} = |\psi^{(i)}\rangle\langle\psi^{(i)}|$, which its trace over the momentum gives us a reduced density matrix for the spin, that is

$$(\varrho^{(i)})_{\sigma'\sigma} = \int d^3p \langle p, \sigma'|\rho^{(i)}|p, \sigma\rangle = \int d^3p C_\sigma(p) C^*_{\sigma'}(p) \tag{39}$$

or

$$\varrho^{(i)} = \int d^3p C(p) C^\dagger(p). \tag{40}$$

Also, we can find the final reduced density matrix by taking the trace of $\rho^{(f)} = |\psi^{(f)}\rangle\langle\psi^{(f)}|$ over the momentum, that is

$$\varrho^{(f)}_{\sigma'\sigma} = \sum_{\sigma''\sigma'''} \int d^3p C_{\sigma''}(p^a) C^*_{\sigma'''}(p^a) \times D_{\sigma'\sigma''}(W(\Lambda(x_f, x_i), p)) D^*_{\sigma\sigma'''}(W(\Lambda(x_f, x_i), p)). \tag{41}$$

This can be written in a more compact form as

$$\varrho^{(f)} = \int d^3p [D(W, p) C(p)][D(W, p) C(p)]^\dagger. \tag{42}$$

It must be noted that though both the states, $|\psi^{(i)}\rangle$ and $|\psi^{(f)}\rangle$, are pure, but the reduced densities $\varrho^{(i)}$ and $\varrho^{(f)}$ are generally mixed. Even if in a special case $\varrho^{(i)}$ is pure (for example in (40) let $C_\downarrow(p) = 0$), the final reduced density will be mixed, as we see from (41). Then, to calculate the spin fidelity we use the relation

$$\mathcal{F}_s = \mathrm{tr}(\varrho^{(i)} \varrho^{(f)}) + 2\sqrt{\det\varrho^{(i)} \cdot \det\varrho^{(f)}}, \tag{43}$$

which is valid only in 2 dimensions [6].

For the circular motion around the throat, first note that (31) vanishes for the orbits closing the throat where $B(r)=r$. Then the matrix $D(W,p)$ becomes the unit matrix and (42) reduces to (40). This means, for circular orbits coinciding the throat, the initial reduced density matrix remains intact and we have $\mathcal{F}_s = \mathrm{tr}({\varrho^{(i)}}^2) + 2\det(\varrho^{(i)})$ which identically equals 1.

One may argue that, Θ vanishes again, if the shape function satisfies the equation $\Phi'(r) = \frac{1}{r}$. But, it can be shown that this violates the asymptotic flatness of the metric.

Now let us choose $C = \binom{F(p)}{0}$. Then

$$\varrho^{(i)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{44}$$

which apparently is a pure density matrix, and

$$\varrho^{(f)} = \begin{pmatrix} \overline{|D_{\uparrow\uparrow}|^2} & \overline{D_{\uparrow\uparrow}D^*_{\downarrow\uparrow}} \\ \overline{D_{\downarrow\uparrow}D^*_{\uparrow\uparrow}} & \overline{|D_{\downarrow\uparrow}|^2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+\overline{\cos\Theta} & -\overline{\sin\Theta} \\ -\overline{\sin\Theta} & 1-\overline{\cos\Theta} \end{pmatrix}, \tag{45}$$

where we have used (30) and the overline is defined as $\overline{X} = \int d^3p|F(p^a)|^2 X(p^a)$ which denotes the average over the momentum distribution. Now the fidelity (43) as evaluated for (44) and (45) gives

$$\mathcal{F}_s = \frac{1}{2}(1+\overline{\cos\Theta}). \tag{46}$$

In order to evaluate the averages over the momentum distribution we choose a Gaussian form for $F(p)$, that is

$$F(p) = \frac{\sqrt{\alpha\delta(p^1)\delta(p^2)}}{\sqrt{\sqrt{\pi}mc}} \exp\left[-\frac{\alpha^2(p^3-q^3)^2}{2m^2c^2}\right]. \tag{47}$$

By this choice we reach to

$$\overline{\cos\Theta} = \frac{\alpha}{\sqrt{\pi}} \int dp e^{-\alpha^2(p-q)^2} \cos\Theta, \tag{48}$$

where $p = \frac{p^3}{mc}$. Evidently, $\overline{\cos\Theta}$ vanishes as $q\to\infty$, resulting $\mathcal{F}_s = \frac{1}{2}$.

To follow the argument more precisely, we choose a constant red shift function as $\Phi(r) = \Phi_0$ and a shape function as $B(r) = \frac{r_0^n}{r^{n-1}}$ where n is a positive integer and r_0 denotes the radius of the throat. Then (31) reduces to

$$\Theta = 2\pi\left(\frac{\tau}{\tau_0}\right)\sqrt{\frac{1}{s^2}-\frac{1}{s^{n+2}}}\left[q(q^2+1) - q^2\sqrt{q^2+1}\frac{p}{\sqrt{p^2+1}+1}\right], \tag{49}$$

where τ_0 is the proper time when a photon rotates once on a circle just coinciding the throat and $s = \frac{r}{r_0} \geq 1$. By this Θ still there is no analytical solution for the integral (48). However, some important results can be attained from a numerical approach. In Fig. 2, the spin fidelity given by (46) is sketched versus a varying q but fixed τ and s. This curve, which is obtained numerically, follows a damped fluctuation such that for $q\to\infty$ it approaches to the average value $\frac{1}{2}$. It must be noted that significant fluctuations occur for small q's. Because, as q grows the number of turns of the centroid around the throat increases. This leads eventually to a saturation in the spin fidelity.

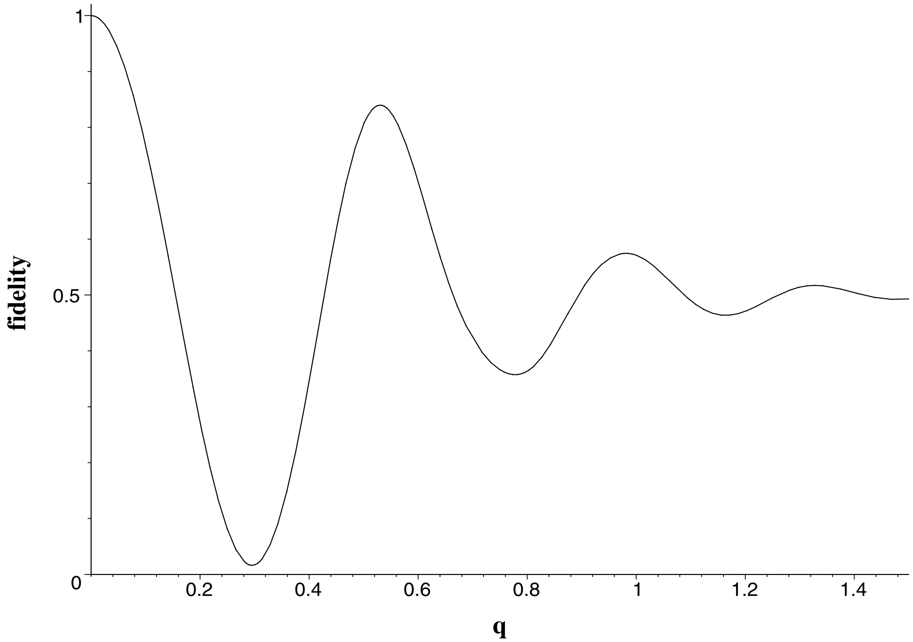

Fig. 2 A sketch of $\mathcal{F}_s$ for fixed s and τ but varying q. Significant fluctuations occur for small values of q, while as q goes to large values the fidelity reaches to a saturated value $\frac{1}{2}$

More interesting here is to study the fidelity (46) for a varying radius r but fixed q and τ. Figure 3 shows the corresponding sketch. Note that s $(=\frac{r}{r_0})$ on the horizontal axis begins with 1 which denotes the throat. As the figure shows, for circular paths coinciding the throat we have a perfect spin fidelity $\mathcal{F}_s = 1$, which can lead to an accurate quantum information transmission. On the other hand, as a consequence of asymptotic flatness of wormhole, as $s \to \infty$ we again have a perfect fidelity. It is remarkable that there exists one minimum in the fidelity for a certain circular path with radius r_m given by

$$r_m = s_m r_0 = \left(\frac{n+2}{2}\right)^{\frac{1}{n}} r_0, \tag{50}$$

where s_m denotes the minimum and is obtained naturally as the root of the derivative of (49) with respect to s. In Fig. 3 we have chosen $n = 2$, hence $s_m = \sqrt{2}$.

Finally, for the radial motion through the throat $D_{\sigma'\sigma}(W(x)) = \delta_{\sigma'\sigma}$, and so the density matrix remains intact, as is seen from (41). Thus $\mathcal{F}_s$ in (43) identically equals 1. Such a result will be correct whether the red shift function is constant or not.

5 Conclusion

We generally proved that both acceleration and gravitation lead to a spin decoherence when the centroid of the wave packet of spin-$\frac{1}{2}$ moves on a specified path in a gravitational field. Then we applied the result for a static spherically symmetric traversable wormhole. Absence of any event horizon for wormholes considered and their non-trivial topology which allow

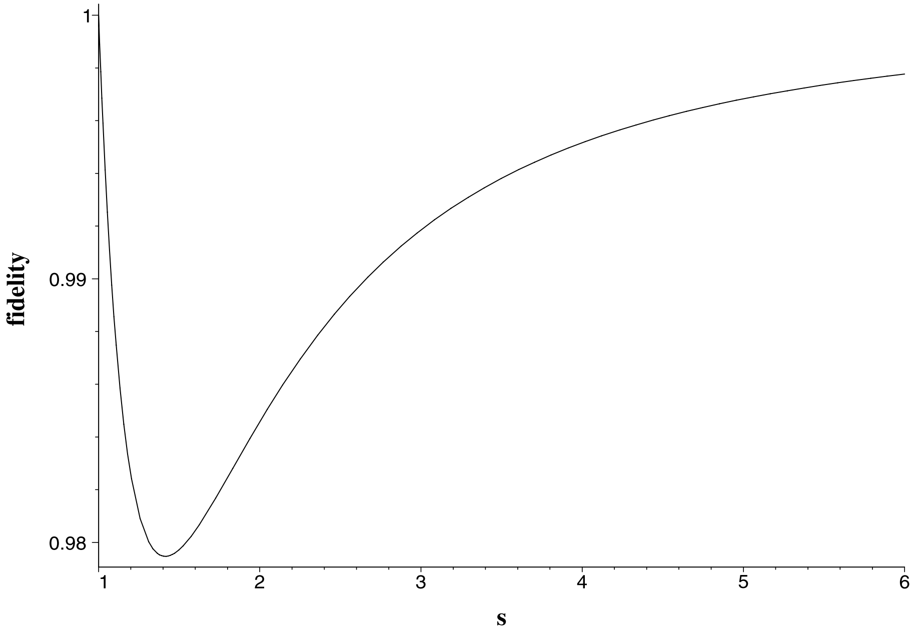

Fig. 3 A sketch of $\mathcal{F}_s$ for fixed q and τ but varying s. Just at the throat ($s = 1$) the fidelity equals 1, suggesting a perfect communication. For wormholes considered there exists one circle on which $\mathcal{F}_s$ is minimum. Here n is chosen to be 2, then the minimum occurs at $r_m = \sqrt{2}r_0$. As a consequence of asymptotic flatness of wormhole, as $s \to \infty$, $\mathcal{F}_s \to 1$

them to be traversable, motivate us to consider wormholes here, although some theoretical and experimental problems exist with them. We discussed circular forced motions and radial geodesic motions of the centroid, separately. To measure the occurred spin decoherence, we used the concept of fidelity which is a basic component in communication theory. We discussed both the total fidelity and the spin fidelity between the initial an final states. Our calculations showed that for circular paths coinciding the throat of wormhole the total fidelity as well as the spin fidelity are perfect. This means that error in quantum communication diminishes along a circular path connecting two observers situated at the throat. We proved that the spin fidelity depends on the angular momentum of the centroid, the proper time during which the motion occurs and the radius of the path. For fixed proper time and radius, as angular momentum grows the number of turns of the centroid around the throat increases, then the spin fidelity reaches to a saturated value. While for fixed angular momentum and proper time, as the radius grows from the throat to the infinity, the spin fidelity begins with the unity at the throat, then takes a minimum value at a determined radius and finally returns to the unity far from the throat. Moreover, our calculations showed that there is no spin decoherence for radial motions. Then in this case the spin fidelity always equals to the unity. This means that quantum communication via the spin parts of the states can occur perfectly along a radial geodesic connecting two observers each of them located in one of the sides of wormhole. Of course, in this case the total fidelity is not generally perfect.

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